# Who Proved Pythagoras's Theorem?

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*Socrates: Try to tell me then how long a line you say it is. — Three feet.*
—Plato, *Meno*, 83 e.

The initial proof of the Pythagorean theorem was neither about equality by complementation such as Euclid's proposition I.47 nor based on Eudoxus's proportion theory like VI.31. It must have been about lengths and areas, but that could have happened only before the notion of length was abandoned from geometry after Eudoxus introduced a new definition of proportionality.

In the *Elements*, Euclid proves the Pythagorean theorem two times, in propositions I.47 and VI.31. In both proofs, he refers to the equality of a square on the leg of a right triangle and the rectangle contained by the hypotenuse and the orthogonal projection of the leg to the hypotenuse (Figure 1).

The proof of I.47 relies on I.35 and I.41, but the notion of *equality*, which is essential for the understanding of the Pythagorean theorem, in propositions I.35 and I.41 has two different meanings. That makes Euclid's use of this notion inconsistent. The presence of inconsistency indicates that the proof of I.47 is possibly a result of the improvement of an older proof of the Pythagorean theorem, one that unifies two different meanings of the notion of equality.

The proof of the equality of a square and a rectangle in VI.31 is placed in the context of Eudoxus's proportion theory, which is expounded in Euclid's Book V.[1] The notion of equality used in VI.31 differs from the notions used in I.35 and I.41. What harmonizes these three different understandings of the notion of equality is that all of them lead to the same conclusion, that the area of the square on a hypotenuse is equal to the sum of the areas of the squares on the legs. This is what indicates that the initial proof of the theorem that is attributed to Pythagoras was about the equality of areas of squares.

Placing the initial proof of the Pythagorean theorem in the early Pythagorean period serves as a starting point for the reconstruction of the chronology of events that are of importance for the early history of the Pythagorean theorem.

Although not mathematicians, Plato and Aristotle are in the chronological chart of Figure 2 because they witnessed and commented on progress in geometry caused by Eudoxus's new approach to proportion theory.

The present paper intends to underscore the importance of Eudoxus in shaping the methods of Euclid's *Elements* and to delineate the different forms of equality in Euclid as well as his inconsistent uses of the term. It will be pointed out that what was new about Eudoxus's proportion theory was not its applicability in the cases of commensurable and incommensurable magnitudes—for the pre-Eudoxian definition of proportionality was also applicable in both cases—but its independence from the notion of length.

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[1] According to a scholium attributed to Proclus, proportion theory from Euclid's Book V is the work of Eudoxus [5, vol. 2, p. 112].




https://doi.org/10.1007/s00283-022-10205-x

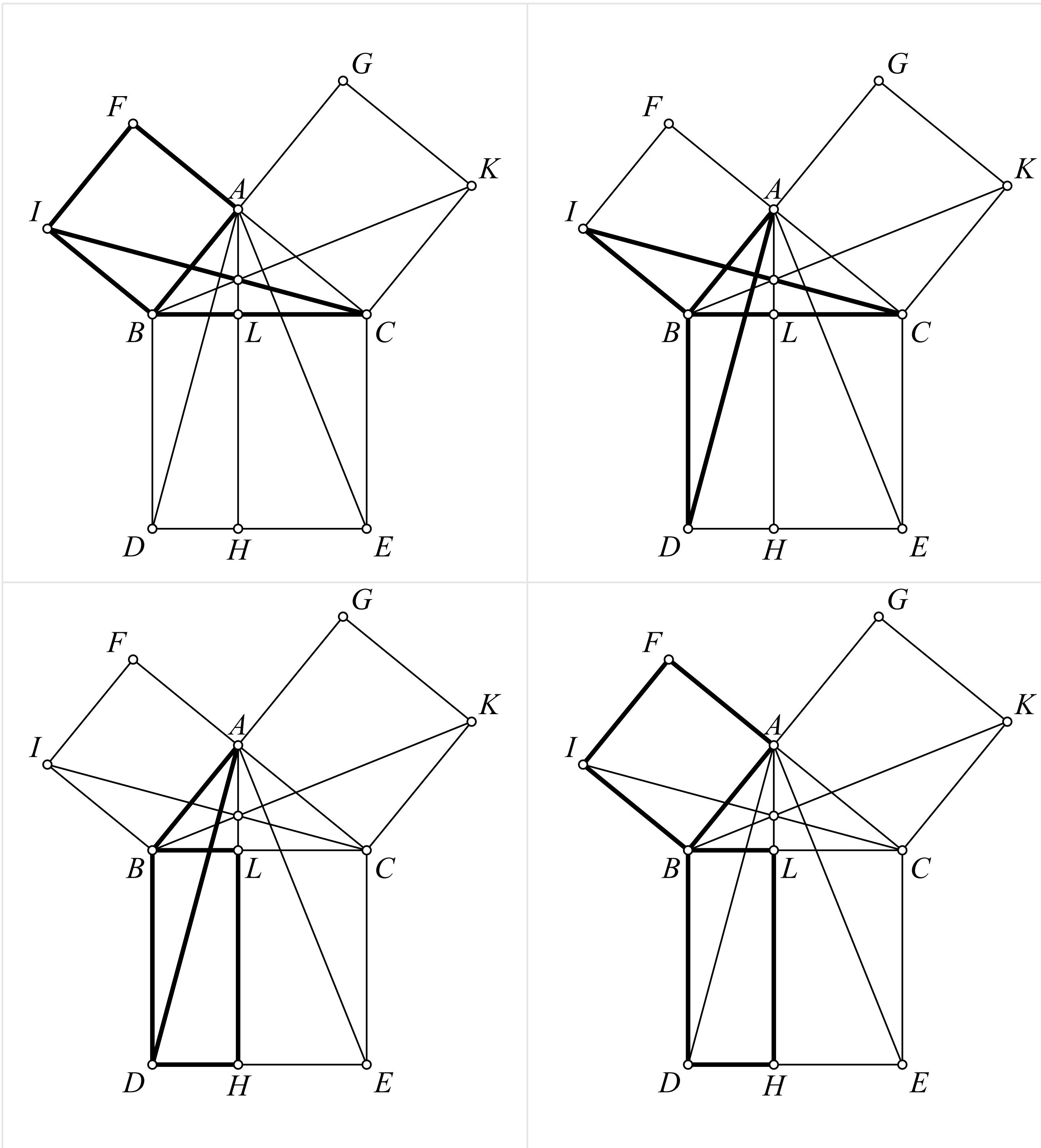


**Figure 1.** The *Elements*, I.47: $IBAF = 2IBC = 2BDA = BDHL$.

As will be shown, neither the proof of VI.31, which is based on Eudoxus's proportion theory, nor the proof of I.47, which looks elementary because it is about "visible" features of plane figures, originates from the time before Eudoxus established his theory of proportions. It will be argued that the initial proof of the Pythagorean theorem could not have been about sides of right triangles and squares on them as Euclid's proofs are, but about their measures—lengths and areas. This means that the Pythagorean theorem was initially about the *Pythagorean equality* $a^2 + b^2 = c^2$. Therefore, it was proved as a proposition of "geometric algebra."[2]

[2]For the concept of geometric algebra, see [8] and [10, pp. 34–48].

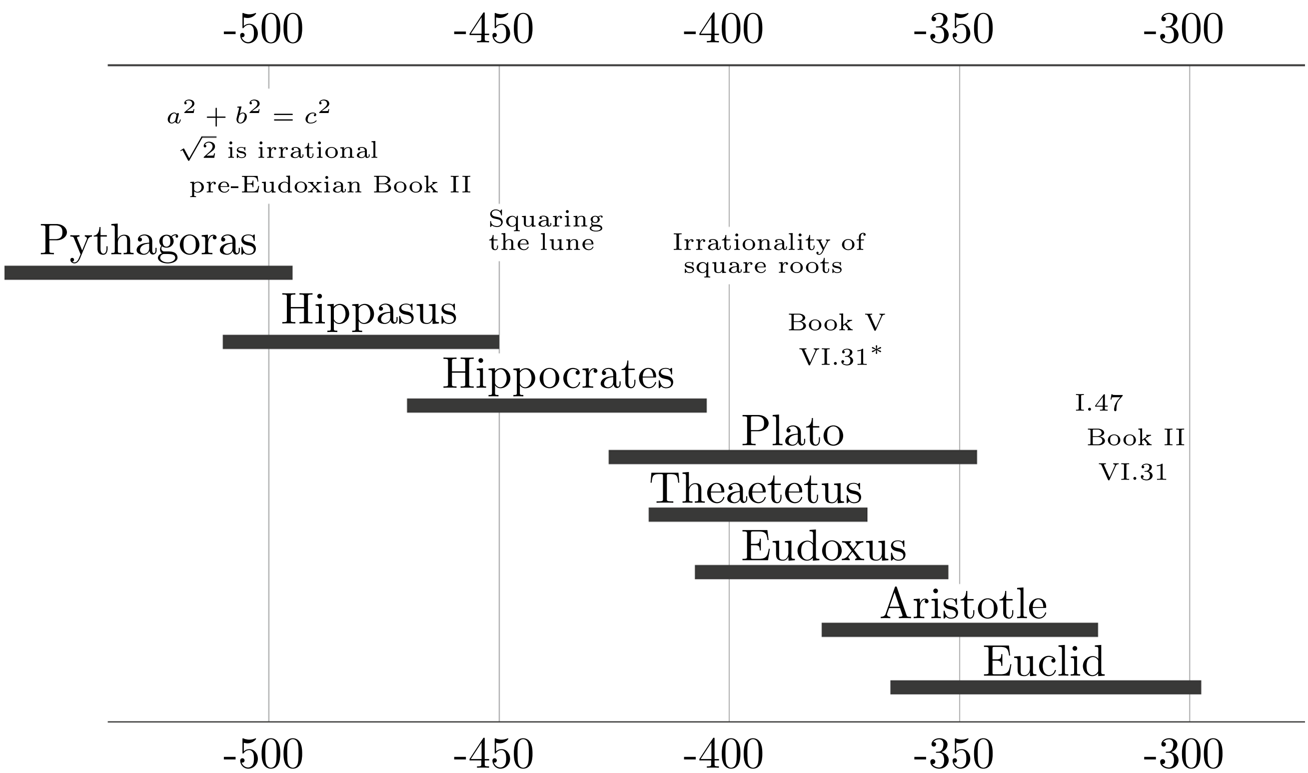


**Figure 2.** Chronology chart for the Pythagorean theorem.

Euclid's approach to the Pythagorean theorem is analyzed in the first three sections below: "Pythagorean Triples," "A Most Lucid Demonstration," and "Proposition VI.31*." The first of these is a brief display of arithmetic aspects of the Pythagorean theorem that are examined in Book X of the *Elements*. The proof of Euclid's proposition I.47, which is about equality by complementation, is analyzed in the next section. Euclid's proposition VI.31, whose proof is based on Eudoxian proportion theory, is the focus of the third section. The differences between the notions of equality in these two propositions are underlined in the section "What Does Equal Mean?" while the next section is about Hippocrates's solution to the problem of the quadrature of the lune. We argue that the solution is about equality of areas of the lune and the triangle. That the Pythagorean theorem must have been about equality of areas of the squares is argued in the following section.

As shown in the section "Aristotle's Definition," the definition of proportionality that had been in use before Eudoxus was about rectangular areas and lengths of their sides. That the Pythagorean theorem in the pre-Eudoxian period could only have been proved as a theorem of "geometric algebra" is shown in the section "Pythagoras's Defective Proof." We argue that Euclid's proposition VI.31 in the special case in which figures on the sides of a right triangle are squares is just an improved version of the early-Pythagorean proof.

That the proof of Euclid's proposition I.47 was also inspired by the early-Pythagorean proof of the Pythagorean theorem is argued in the section "Euclid's Harmonization."

## Pythagorean Triples

Before the discovery of incommensurability, the Pythagorean theorem could only have been related to right triangles with sides of integer length.[3] That is, it could have been only about *Pythagorean triples* $(p, q, r)$ of whole numbers $p, q, r$, such that $p^2 + q^2 = r^2$.

The Greeks recognized a general method for finding such triples.[4] However, they were familiar with triples $(a, b, c)$ of not necessarily rational numbers that also satisfy the *Pythagorean equality* $a^2 + b^2 = c^2$. It was known to them that the triple $(1, 1, \sqrt{2})$, which characterizes isosceles right triangles, satisfies this equality.[5] Judging by

[3]The case of rational side lengths is easily reduced to the case of sides of integer length. It is merely a matter of the selection of a segment of unit length.

[4]In lemma 1 to proposition X.28 of the *Elements*, Euclid records a method for generating Pythagorean triples. Proclus discusses this issue in [14, p. 340].

[5]In Plato's *Meno* (82b–85b), Socrates leads young Meno's slave to the conclusion that the square whose side is a diagonal of a given square is twice as large as that square.

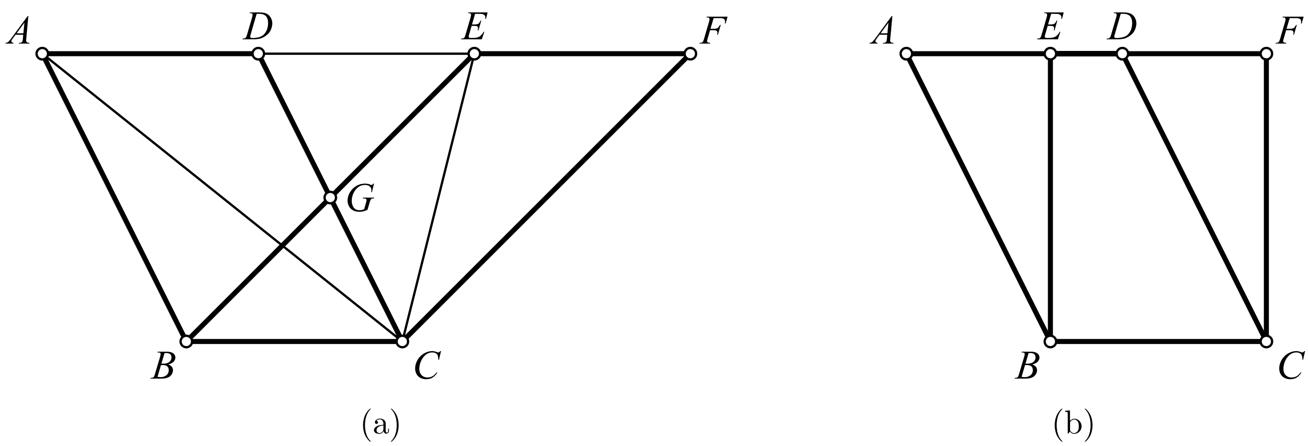


**Figure 3.** The *Elements*, I.35 and I.41.

the content of Book X of Euclid's *Elements*, in the time of Theaetetus, it was known that every triple of the form $(\sqrt{p}, \sqrt{q}, \sqrt{r})$ such that $p + q = r$, where $p$, $q$, and $r$ are natural numbers, also satisfies the Pythagorean equality.[6]

If $p$, $q$, and $r$ are perfect squares, then $(\sqrt{p}, \sqrt{q}, \sqrt{r})$ is obviously a Pythagorean triple. If at least one of them is not a square, then the triangle characterized by the triple of the form $(\sqrt{p}, \sqrt{q}, \sqrt{r})$ can serve in proving the incommensurability of its sides.[7] As in the case of Pythagorean triples, triples of this form must have been considered triples of side lengths of right triangles.

It seems that in the time of Euclid, very few examples of triples that satisfy the Pythagorean equality not of the form $(p, q, r)$ or $(\sqrt{p}, \sqrt{q}, \sqrt{r})$ could have been known. That is because in the general case, triples $(a, b, c)$ are formed by the lengths $a$, $b$, $c$ of sides of right triangles, but an understanding of the notion of *length of a line segment* had not been reached by Greek mathematics.[8]

## A "Most Lucid Demonstration"

The Pythagorean theorem in some form was known and applied before the time of Pythagoras.[9] However, Proclus informs us that he admired "those who first became acquainted with the truth of this theorem," but he also adds that he "marvel[s] more at the writer of the *Elements*, not only because he established it by a most lucid demonstration, but because he insisted on the more general theorem by the irrefutable arguments of science in the sixth book" [15, vol. I, p. 185]. Therefore, Proclus credited Euclid for establishing the Pythagorean theorem by a most lucid proof (*Elements*, I.47) and for the irrefutable demonstration of its generalization (VI.31).

In proposition I.47, Euclid proves the following:

> In right triangles, the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Depending on what is perceived as being the "square on the side" and what is the understanding of the "sum of squares," this formulation of the Pythagorean theorem could be comprehended in at least three different ways.

1. If squares on the sides of a right triangle are considered squares of their lengths, say $a^2$, $b^2$, $c^2$, thus three real numbers, then the content of I.47 can easily be expressed by the formula $a^2 + b^2 = c^2$.
2. The same formula of "geometric algebra" holds if $a^2$, $b^2$, and $c^2$ are considered areas of the squares.[10] However, this formula is not what Euclid proves in I.47 or anywhere else in the *Elements*.
3. What Euclid proves in I.47 is that the square on the hypotenuse of a right triangle and the squares on the legs are *equicomplementable*, or *equal by complementation*.[11] This means that the square on the hypotenuse on the one hand, and the squares on the legs on the other, can be complemented by two mutually congruent (or equidecomposable) figures such that the union of the square on the hypotenuse with the first figure is *equidecomposable* with the union of squares on the legs with the second. That

[6] Book X is about such triples. Judging by Euclid's definitions X. def. III.1–6, an apotome is the difference of the hypotenuse and a leg of a right triangle determined by a triple that is of the form $(\sqrt{p}, \sqrt{q}, \sqrt{r})$. Binomials (X. def. II.1–6) are their sums.

[7] For many examples of such triangles, see [11, pp. 314–327].

[8] For the definition of the measure of a segment, see [3, pp. 167–172].

[9] Van der Waerden believed that in the Neolithic age there must have been an established doctrine of Pythagorean triples and their ritual applications [17, p. 25].

[10] Because if $a$, $b$, $c$ are the lengths of the sides, then $a^2$, $b^2$, $c^2$ are the areas of the squares on them [12, p. 166].

[11] For the notions of equidecomposability and equicomplementability, see [7, §18].

> they are equidecomposable, or *equal by decomposition*, means that each of the two unions can be decomposed into a finite set of "smaller" disjoint figures that are in one-to-one correspondence so that the corresponding pairs of the smaller figures are congruent.

Euclid does not define the equality of figures either by decomposition or by complementation. If figures are equidecomposable or equicomplementable, he simply calls them *equal*. A part of Book I of Euclid's *Elements* and all of Book II are about equal figures. The first such proposition in the *Elements* is I.35, in which it is proved that two parallelograms with the same base and contained between the same two parallels are equal. Finding that such parallelograms $ABCD$ and $EBCF$ have the triangle $GBC$ in common (Figure 3a), while quadrangles $ABGD$ and $EGCF$ that complement this triangle to form the parallelograms $ABCD$ and $EBCF$ are obtained by "subtraction" of the triangle $DGE$ from two congruent triangles $ABE$ and $DCF$, Euclid proves that $ABCD$ and $EBCF$ are equal by complementation.

Euclid does not discuss the case of equidecomposability of $ABCD$ and $EBCF$; that is, he ignores the possibility that segments $CD$ and $BE$ could have empty intersection (Figure 3b). He does not prove that in this case, having equal parts, the parallelograms $ABCD$ and $EBCF$ are equal by decomposition. He ignores the case of equidecomposability because in the proof of the Pythagorean theorem, I.47, only the case of equicomplementability matters.

By I.41, triangles $ABC$ and $EBC$ are halves of the equal parallelograms $ABCD$ and $EBCF$. Consequently, by Common Notion 2, they are also equal.[12] However, Euclid does not go into detail to determine whether they are equal by decomposition or complementation.[13] This shows an inconsistency in Euclid's understanding of the notion of *equality*.

Since it is based on I.35 and I.41, Euclid's proof of I.47 is about equality by complementation. What Euclid finds in the proof of this proposition is that each of the squares on the legs is equal by complementation to the rectangle contained by the hypotenuse and the orthogonal projection of the leg to the hypotenuse.

If $IBAF$ is the square on the leg $AB$ of a right triangle $ABC$ (Figure 1), $BDHL$ the rectangle contained by $BD$, which is equal to the hypotenuse, and $BL$, which is the orthogonal projection of $AB$ on the hypotenuse $BC$, then $IBAF = 2IBC = 2BDA = BDHL$. Similarly, the square $GACK$ on the leg $AC$ is equal by complementation to the rectangle $LHEC$. Therefore,

$$IBAF + GACK = BDHL + LHEC = BDEC,$$

which proves the Pythagorean theorem.

## Proposition VI.31*

Euclid's proposition VI.31 is more general than I.47. It is about the equality not only of squares but of any mutually similar figures on the sides of a right triangle. In VI.31, Euclid proves the following:

> In right triangles, the figure on the side opposite the right angle equals the sum of the similar and similarly described figures on the sides containing the right angle.

The proof relies on Eudoxus's proportion theory, which is about the proportionality of line segments and polygonal regions, and not about the proportionality of lengths of segments and areas of polygons, although it may look so to a modern reader.

In the case that similar figures on the sides of a right triangle are squares, proposition VI.31 by its formulation takes the form of I.47. As a special case of VI.31, the proposition in this form will be denoted by VI.31*. The proof of proposition VI.31 is based on Euclid's theory of similitude from Book VI, namely, on proposition VI.8 and the porism of VI.19, but in the case of VI.31*, the proof relies on VI.17 instead of on the porism of VI.19 [6, vol. I, p. 378].

## What Does Equal Mean?

There is a common point in the proof of proposition I.47 on the one hand and propositions VI.31 and VI.31* on the other. They refer to the rectangle contained by the hypotenuse and the orthogonal projection of a leg to the hypotenuse. In I.47, that reference is explicit, while in VI.31 and VI.31*, it is in the context of Eudoxian proportion theory. Furthermore, propositions I.47 and VI.31* are about *equality* of the rectangle and the square on the leg. In the case of I.47, that is equality by complementation. In VI.31*, it is not. It is about the equality of parallelograms that is derived from the *sameness* of ratios of their sides.[14] In the case of I.47, the notion of equality looks "elementary," since it is about "visible" features of plane figures. In the case of VI.31*, it is far from being elementary, since it depends on Eudoxus's proportion theory.

Understanding proportions as being about line segments and polygonal regions as in VI.31 demands a high level of abstraction. Such understanding became possible only after Eudoxus founded his theory of proportions. In the time between Thales and Eudoxus, proportions were used in solving various geometric and practical problems, but the notion of proportion was defined differently from how it appears in Eudoxus's theory. A good example of the use of a pre-Eudoxian notion of proportionality is Hippocrates's famous solution to the problem of the quadrature of the lune.

[12] In some manuscripts of the *Elements*, among Common Notions there is one that could be applied directly at this place. That is: "Things that are halves of the same thing are equal to each other" [5, vol. I, p. 223].

[13] For a proof that two triangles are equicomplementable if they have equal bases and equal altitudes, see [7, §19, Th. 27].

[14] Propositions VI.14–17 are about the relation between equality of parallelograms or triangles and the sameness of ratios of their sides.

## Quadrature of the Lune

By its formulation, proposition VI.31 is about arbitrary similar figures, but its proof is about rectilinear similar figures. However, that it is valid also for some curvilinear figures was known at least a hundred years before Euclid.

In his answer to the question whether a curvilinear figure, the lune, could be equal to a rectilinear figure, the triangle, Hippocrates of Chios explains that "similar segments of circles have the same ratios as the squares on their bases."[15] He circumscribes a semicircle about a right isosceles triangle $ABC$ (Figure 4) and a circular segment about its base $AB$ that is similar to those cut off by its sides $AC$ and $CB$. Using the Pythagorean theorem, he found that the circular segment about $AB$ is equal to the sum of the segments about $AC$ and $CB$. Consequently, since the lune $ABCA$ that is enclosed by the semicircle on $AB$ and the circular segment $AB$ is the "sum" of the part of the triangle "above" the circular segment $AB$ and the circular segments about $AC$ and $CB$, the lune is equal to the triangle $ABC$.

At first sight, it may appear that the triangle and the lune are equal by decomposition, since they consist of equal parts. However, the circular segment about the base $AB$ is not equal by finite decomposition to the sum of the segments about the sides $AC$ and $CB$. They are equal in the context of pre-Eudoxian proportion theory. This gives the impression that in Hippocrates's solution to the problem of the quadrature of the lune, the word *equal* has two different meanings. One is about equality by decomposition, so it is similar to the meaning that is used in the proof of I.47. The other is about equality in the context of proportions, so its meaning is similar to that from the proof of VI.31. Both meanings are implied in the solution of the problem. What harmonizes these two meanings is that both lead to the same conclusion about the equality of areas.

Since it is simultaneously about equidecomposability and equality in the context of proportion theory, Hippocrates's solution in effect is about equality of areas of the lune and the triangle. Therefore, the Pythagorean theorem that he applies in the solution is about the equality of the area of the square on the hypotenuse of a right triangle and the sum of the areas of the squares on the legs, and thus a different form of the theorem from what is proved in the *Elements*.

## Area and Length

With the adoption of a new definition of proportionality, Eudoxus enabled the possibility of leaving the notion of area out of proportion theory. Euclid embraced this possibility in the *Elements*. However, he found that even without a definition of area, it is possible to realize that two equidecomposable or equicomplementable figures are of equal areas. Two figures considered as magnitudes, as in

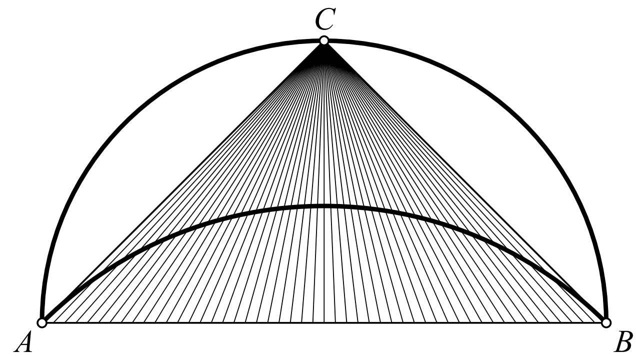


**Figure 4.** The lune $ABCA$ is equal to the triangle $ABC$.

VI.31*, are of equal area if they are equal in the context of proportion theory.

Instead of finding a way to define the notion of area as a measure of a plane figure,[16] Euclid used the fact that however *area* might be defined, it must satisfy the property of finite additivity. This is what his proof of I.47 is about. However, this is not the case with VI.31*. Its proof is based on a much more sophisticated theory than I.47, but paradoxically, it seems that the proof of VI.31*, carried out by a simple use of proportionality of sides of similar triangles, inspired the proof of I.47 [13, p. 172].

Since according to Proclus, it was Euclid who established I.47 "by a most lucid demonstration," we may infer that before Euclid, none of the proofs of the Pythagorean theorem was carried out by the use of equality by decomposition or complementation.[17] If there had been such a proof, the theorem would be assessed as having been equally as well established as Euclid's proposition I.47, but in the pre-Euclidean period. Such a proof would be considered equally as "lucid" as Euclid's proof of I.47. But in the preserved ancient literature, there are no traces of such a proof before the time of Euclid.

Since it was based on Eudoxian proportion theory, the proof of VI.31 was not possible before Eudoxus. Proclus provides further information, claiming that the proof is Euclid's. However, this does not necessarily mean that the proof of VI.31* is also Euclid's. Proclus does not claim that Euclid proved VI.31* but that he insisted on a more general theorem, that is, VI.31.

Since Euclid's arguments used in the proof of VI.31 are assessed by Proclus as irrefutable, this could mean that for some reason, arguments that were used by "those who first become acquainted with the truth of this theorem" were considered refutable. Judging by Hippocrates's solution to the problem of the quadrature of the lune, this could be only because the pre-Eudoxian proof of the Pythagorean theorem was grounded on the notions of length and area, thereby rendering the proof incomplete.

[15] For Eudemus's report on the answer to the question, see [15, vol. I, pp. 235–239], [6, vol. I, pp. 191–192], and [9, pp. 163–167].

[16] This notion was far beyond the reach of Greek mathematics. For the definition of the area function and areas of polygonal regions, see [12, pp. 153–175].

[17] For many different visually obvious proofs of the theorem, see [5, vol. I, pp. 364–366] and [6, vol. I, p. 149].

## Aristotle's Definition

The definition of proportionality that was in use before Eudoxus is suggested by Aristotle:

> The line parallel to the side and *cutting* the plane figure *divides* similarly the base and the area ... for the areas have the same *antanairesis* as have the sides: and this is the definition of *the same ratio* (Aristotle, *Topics*, 158b).

Aristotle's definition is about *plane figures*, but Alexander clarifies that Aristotle's plane figures are parallelograms [11, p. 258].

Aristotle explains that a line parallel to its lateral side cuts a parallelogram into two parallelograms, and its base into two line segments, such that the parallelograms and the segments have the same *antanairesis*, which means that they are in the same ratio. Thus what Aristotle defines is the proportionality of line segments and parallelogram regions.

Intending to derive Aristotle's definition from definition V. def. 5, in proposition VI.1, Euclid proves that if $B$, $C$, $D$ are three collinear points, then the ratio of line $CB$ to line $CD$, the ratio of triangle $ACB$ to triangle $ACD$, and the ratio of the parallelogram $ACBE$ to parallelogram $ACDF$ are all the same ratio (Figure 5). Thus although lines are not of the same kind of magnitudes as triangles and parallelograms, their ratios still can be compared to determine whether such ratios are the same. They can be compared without the need of measuring lengths of lines and areas of triangles or parallelograms.

In Book VI, Euclid very rarely applies Eudoxus's definition directly. Except for proposition VI.33, the entire Book VI depends directly not on V. def. 5, but its consequence, VI.1. So the role of proposition VI.1 is exceptional. Euclid placed it at the very beginning of Book VI, aiming to prove first that Aristotle's definition of proportionality is a consequence of V. def. 5 [1]. Then, after proving VI.1, he could hold on to the pre-Eudoxian proofs of the propositions on similitude, those that rely on Aristotle's definition.

Since Eudoxus's definition is applicable to incommensurable as well as commensurable line segments, so is Aristotle's definition. Even if Aristotle's definition is recognized as being about lengths of sides of parallelograms and their areas, it is again applicable in cases of commensurable and incommensurable line segments. Indeed, if lines $BC$, $CD$, $EB$ are supposed to be commensurable (Figure 6a), then their greatest common measure could be considered to be of unit length. Therefore, Aristotle's definition, in this case, is about lines of integer lengths $r$, $s$, $k$, and rectangles of integer areas $kr$ and $ks$, so it could be expressed in terms of number theory:

> plain numbers $kr$ and $ks$ are proportional to their sides $r$ and $s$.

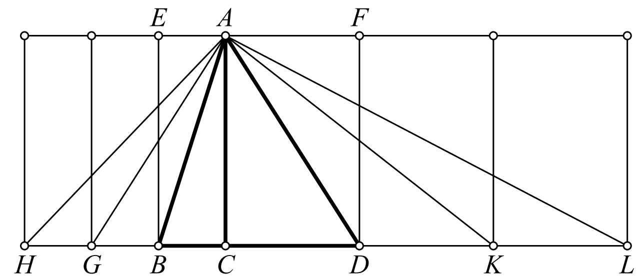


**Figure 5.** The *Elements*, VI.1.

This is what Euclid proves in VII.17. But if lines $BC$, $CD$, and $EB$ are supposed to be three incommensurable lines of lengths $a$, $b$, $h$ (Figure 6b), then a simple generalization of VII.17 based on the presupposed existence of the functions of length and area gives that

> rectangular areas $ha$ and $hb$ are proportional to the lengths $a$ and $b$ of their sides.

This is the form of Aristotle's definition that could have been in use in the pre-Eudoxian period. In this form, the definition of proportionality is applicable to Hippocrates's solution.

## Pythagoras's Defective Proof

Hippocrates applies a version of the Pythagorean theorem that is about the equality of areas. In this light, the Pythagorean theorem initially could only have been proved as a theorem of "geometric algebra." If pre-Eudoxian "geometric algebra" is just fiction,[18] Hippocrates's solution to the problem of the quadrature of the lune is unexplainable.

In the time of the early Pythagoreans, the proof of the Pythagorean theorem was about lengths and areas. The truth that $a^2 = b^2 + c^2$ if $a$, $b$, $c$ are lengths of the hypotenuse and the legs of a right triangle $ABC$ appears as a consequence of the similarity of the triangle $ABC$ with triangles $DBA$ and $DAC$ obtained by decomposition of $ABC$ by its height $AD$ (Figure 7). Indeed, if $BD$ and $CD$ are of lengths $k$ and $l$, then from the proportionalities $a : b = b : l$ and $a : c = c : k$, it follows that $b^2 = a \cdot l$ and $c^2 = a \cdot k$. Consequently, $b^2 + c^2 = a \cdot (l + k) = a^2$, which proves the theorem.[19]

The proof is about the proportionality of lengths of the sides of similar rectilinear figures, so it is grounded on a conviction that the length of every line segment is based on a given unit length. What is missing is the definition of the function of length. Thus the proof is incomplete,[20] but it was within the reach of the early Pythagoreans before Hippasus [18, p. 268].

[18] That "geometric algebra" is a "monstrous concept" that "leads to absurdities" is argued in [16]. That a "geometric algebra" interpretation should be reinstated as a viable historical hypothesis is argued in [2].

[19] Since nobody else has ever been mentioned in ancient literature as a potential author of the theorem, there is no reason to doubt that the early Pythagorean proof is Pythagoras's.

[20] For a complete proof, see [3, pp. 167–176, 273].

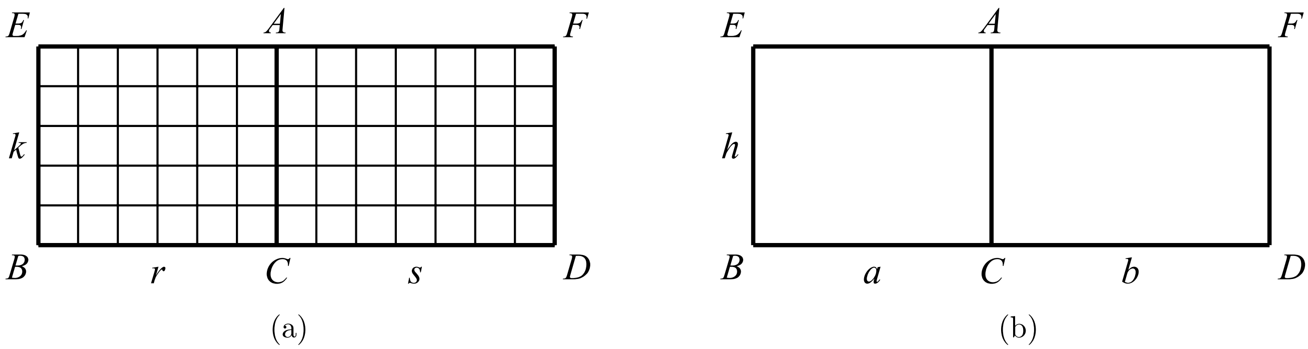


**Figure 6.** The *Elements*, VII.17 and pre-Eudoxian VI.1.

The difference between this proof and the proof of VI.31* is the definition of proportionality that is applied. The defective proof applies Aristotle's definition in its pre-Eudoxian form, while the proof of VI.31* applies Eudoxus's definition. To that extent, the proof of VI.31* is an improved version of the early Pythagorean proof of Pythagoras's theorem.

The defectiveness of the early Pythagorean proof could have been realized before Eudoxus, but the improvement in the proof was not possible before Eudoxus launched a new approach to proportion theory that freed geometry from the use of notions of length and area. In founding a new theory, he improved early Pythagorean proofs of propositions on proportions, including the proof of the most famous theorem of geometry, Pythagoras's theorem. Eudoxus proved proposition VI.31*. Plato and Aristotle learned from him about the reliable proof of the theorem.[21] Being aware of the defectiveness of the early Pythagorean proportion theory, they simply ignored Pythagoras as a mathematician.

## Euclid's Harmonization

If VI.31* is not about equality of areas of squares, as it is in its pre-Eudoxian proof, what could be the geometric meaning of equality of squares if they are equal in the context of proportions? Proposition I.47 gives a clear answer to the question. Equality of areas of square figures from the early Pythagorean proof is replaced in I.47 by equality by complementation of square regions. This was possible only after Eudoxus launched his theory of proportions and proved VI.31*. Therefore, we can rely on Proclus's words that the author of proposition I.47 is Euclid.[22]

The proof of I.47 was inspired by the early Pythagorean proof of the Pythagorean theorem, but Euclid had to develop the theory of equality by decomposition and complementation, which consists of propositions I.35–45 from Book I, to prove it properly. This new approach to geometry affected the Pythagorean theorem and all of Book II. Together with the early theory of proportions, the method of *application of areas* was initially about lengths and areas.

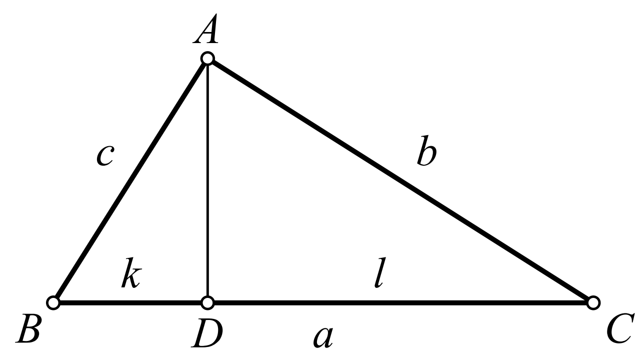


**Figure 7.** Pythagoras's proof.

Euclid reshaped the proofs of the propositions from Book II that were known in the pre-Eudoxian period, intending to place them among the theorems about equidecomposability. Consequently, "geometric algebra" was doomed to be forgotten.

Euclid's purely geometric method employed in the proof of I.47 was not a novelty, since it had been used in Hippocrates's solution. Hippocrates's method looks more like a method about equality by decomposition, while Euclid's proof is about equality by complementation. Guided by the idea that had come from the early Pythagorean proof of Pythagoras's theorem, Euclid missed the fact that the proof of I.47 could be accomplished by the use of equality by decomposition.[23] If he had not missed this, his proof would look even more lucid.

## Summary

Before Eudoxus, geometry was less rigorous than it became after him. The pre-Eudoxian theory of proportions was based on Aristotle's definition of the proportionality of lengths and areas, not that of line segments and plane regions as in the *Elements*. Pythagoras's theorem was also

[21]Diogenes Laërtius informs us that Eudoxus was associated with Plato's Academy [4, 8.86–88].

[22]We could come to the conclusion that the proof was possible only in the period between Eudoxus and Euclid even without Proclus's explanation. Proclus provides further information about the authorship of the proposition.

[23]Such a proof is attributed by an-Nairizi to Thābit ibn Qurra [5, vol. I, pp. 364–365].

about lengths and areas. Initially, it was a theorem of "geometric algebra."

Eudoxus found a way of avoiding the earlier notion of length in the foundations of geometry. This is what his definition of proportionality from Book V is about. On placing this definition in the *Elements*, Euclid left no place in his geometry for the notion of length.

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